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AN ALGEBRAIC APPROACH TO DATA-ADAPTIVE ARRAY PROCESSING.(U)
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AN ALGEBRAIC APPROACH TO DATA-ADAPTIVE ARRAY PROCESSING

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Abstract

In an earlier paper,[1] an algebraic characterization was made of the problem of resolving closely spaced plane waves incident on a linear array. The characterization suggests several data-adaptive processing methods and encompasses the Wiener, Maximum Likelihood, and Pisarenko methods. In this paper, the algebraic approach is amplified and the results extended to consider correlated noise. A recursive algorithm is given for a particularly effective processing method.

Summary

An important array processing problem is that of determining the directions of propagation of plane waves incident on a linear array of uniformly spaced sensors.[2] Contemporary spectral analysis has led to the development of several array processing methods that are able to resolve plane waves with nearly identical directions of propagation. These methods include the Wiener Prediction Filter method,[3] the Maximum Likelihood method,[3] and the Pisarenko method.[4] This paper amplifies and extends an algebraic approach given earlier[1] based upon an algebraic characterization of the array processing problem. The results encompass the methods mentioned above and include the case of correlated noise. A recursive algorithm is presented for implementation of a particularly effective processing method.

Model of the Array Data

Consider the complex sinsoidal time-space plane wave $f(t,\underline{r})$ as represented by

$$f(t,\underline{r}) = Ae^{j[\omega t+\underline{k}\cdot\underline{r}]} \qquad (1)$$

where A is the complex amplitude, t is the continuous time variable, $r = x\hat{i}+y\hat{j}+z\hat{k}$ is the continuous space variable, ω is the (temporal) frequency, and $\underline{k} = k_x\hat{i}+k_y\hat{j}+k_z\hat{k}$ is the wavenumber (spatial frequency). This wave travels in the direction of $-\underline{k}$ with a speed of propagation $c = \frac{\omega}{|\underline{k}|}$. Let us now monitor this wave with a linear array placed along the x-axis whereby $y = z = 0$ as shown in Figure 1. The detected signal is

$$f_x(t,x) = Ae^{j[\omega t+k_x x]} . \qquad (2)$$

From this ideal data it is theoretically possible to *determine the values of the parameters* ω and k_x. Furthermore, if the speed of propagation is a known constant or a known function of frequency, we can then calculate the wavenumber's magnitude from

$$|\underline{k}| = \frac{\omega}{c} . \qquad (3)$$

Because we do not have complete knowledge of the wavenumber $\underline{k}$, we cannot unambiguously determine the direction of propagation. However, we can determine the polar angle γ associated with the wave as

$$\cos\gamma = \frac{k_x}{|\underline{k}|} . \qquad (4)$$

This angle defines a cone whose central axis lies along the linear array. This information alone is sufficient for many applications. For example, the wave may be known *a priori* to be traveling in the xy plane. This is the case we shall consider hereafter.

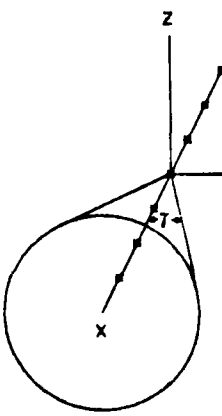

Figure 1. Linear array along x-axis.

We have reduced the problem from four dimensions in the variables to two dimensions, through the constraint of a linear array. We can further reduce the problem to one dimension by noting that time and space are independent quantities in this model. Thus we can perform our analyses for ω and k_x separately. Ordinary time series processing such as filtering or spectral estimation can be applied to each sensor to first determine the presence of signals at a particular temporal frequency ω. These outputs, one for each sensor, are then spatially processed to determine the directions of sources radiating at the frequence ω. Hereafter, we shall suppress the time domain and consider only the spatial dimension.[3,5]

Figure 2 depicts a linear array of p sensors uniformly spaced d units apart. A plane wave is impinging upon the array with an incident angle θ. Noting that the incident angle is complementary to the polar angle, we have

$$\sin\theta = \frac{k_x}{|\underline{k}|} = \frac{k_x}{\omega/c} = \frac{k_x}{2\pi}\lambda$$

from which it is seen that

$$k_x = \frac{2\pi \sin\theta}{\lambda} \qquad (5)$$

where λ is the wavelength of the plane wave. Defining our origin at sensor zero, the nth sensor will sample

the wave at the point x = nd. Hence, at any particular instant in time the array output is, from (2)

$$y(n) = f_x(nd) = Ae^{j\left[\phi + \frac{2\pi \sin\theta}{\lambda} nd\right]} \qquad 0 \le n < P \tag{6}$$

where ϕ is a phase angle dependent on the sampling instant.

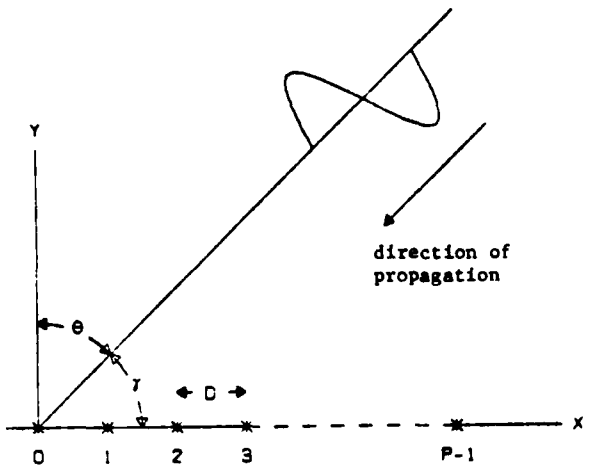

Figure 2. Plane wave with incident angle θ.

The set of p instantaneous spatial samples (6) as measured by the array is referred to as a "snapshot." In this case the snapshot is a sampled complex sinusoid whose sampled spatial frequency is given by $\omega = \frac{2\pi d \sin\theta}{\lambda}$. Clearly, an estimate of the sinusoid's frequency directly yields an estimate of the direction of propagation of the plane wave if the wavelength λ is known. As such, spectral estimation is seen to play a prominent role in linear array processing.

We now generalize our model to include multiple plane waves incident on an array in which the sensor indications are contaminated by white measurement noise. If there are a total of q plane waves and the k^{th} plane wave has a direction of propagation θ_k, it follows by superposition that the snapshot will have the form

$$y(n) = \eta(n) + \sum_{k=1}^{q} A_k e^{j\phi_k} e^{jn\omega_k}, \qquad 0 \le n \le p-1 \tag{7}$$

where the q sinusoid frequencies are given by

$$\omega_k = \frac{2\pi d \sin\theta_k}{\lambda}$$

and $\eta(n)$ are uncorrelated zero mean random variables with variance σ^2 that represent the measurement noise. We assume that the ω_k are all different.

Our objective is to estimate the frequency parameters ω_k using these snapshot measurements. We are particularly interested in the ability to resolve, or distinguish between two plane waves with very similar frequencies (i.e., $\omega_1 \doteq \omega_2$). This estimation capability requires the utilization of a number of snapshots taken sequentially in time. Our data then has the form

$$y_m(n) = \eta_m(n) + \sum_{k=1}^{q} A_k e^{j\phi_{km}} e^{jn\omega_k}, \quad 1 \le m \le M, \; 0 \le n \le p-1 \tag{8}$$

where m is the snapshot index and M is the total number of snapshots. In this model, we assume that the phase angles ϕ_{km} are uncorrelated random variables uniformly distributed on $[-\pi, +\pi]$. Their random nature arises from the independence of the sinusoidal sources and from the approximate randomness of time-sampling far below the Nyquist rate.

It is convenient at this point to represent the problem in vector notation. We represent the m^{th} snapshot (8) by the p×1 column vector

$$\underline{y}_m = [y_m(0) \;\; y_m(1) \;\; \ldots \;\; y_m(p-1)]' \tag{9}$$

and define the pure complex sinusoid vector as

$$\underline{S}_\omega = [1 \;\; e^{j\omega} \;\; e^{j2\omega} \;\; \ldots \;\; e^{j(p-1)\omega}]'. \tag{10}$$

Lastly, the noise vector associated with the m^{th} snapshot is defined as

$$\underline{\eta}_m = [\eta_m(0) \;\; \eta_m(1) \;\; \ldots \;\; \eta_m(p-1)]'. \tag{11}$$

With the above notation, we may compactly represent the snapshots by the data vector equation

$$\underline{y}_m = \underline{\eta}_m + \sum_{k=1}^{q} A_k e^{j\phi_{km}} \underline{S}_{\omega_k}, \quad 1 \le m \le M. \tag{12}$$

The array data (12) is random due to its dependency on the random phase angles ϕ_{km} and the contaminating noise $\eta_m(n)$. Assuming that these random variables are pairwise uncorrelated and statistically invariant with respect to the snapshot index m, it follows that each data vector $\underline{y}_m$ is a windowed realization of a wide-sense stationary random process. The mean value of this process is the zero vector,

$$E\{\underline{y}_m\} = E\{\underline{\eta}_m\} + \sum_{k=1}^{q} A_k \underline{S}_{\omega_k} E\{e^{j\phi_{km}}\} = \underline{0}, \tag{13}$$

while its covariance matrix is specified by

$$R = E\{\underline{y}_m \underline{y}_m^\dagger\} = \sigma^2 I_p + \sum_{k=1}^{q} P_k \underline{S}_{\omega_k} \underline{S}_{\omega_k}^\dagger \tag{14}$$

where I_p is the p×p identity matrix and $P_k = |A_k|^2$ is the power of the k^{th} incident plane wave. Since the random vector process is wide-sense stationary, the covariance matrix R must be positive semi-definite, Toeplitz, and Hermitian.

We now describe three contemporary array processing methods and then present an algebraic approach to identifying the frequencies (ω_k), based upon the structure of the data $\underline{y}_m$ and the associated covariance matrix R.

Contemporary Processing Methods

Wiener Filter Method

The Wiener Filter method is based on filtering the data such that the signal-to-noise ratio at the filter output is maximized.[6] It is essentially a linear prediction approach that is quite similar to the Maximum Entropy method of spectral estimation.[3] Many adaptive array processing algorithms are equivalent to the Wiener Filter method, including Alam's orthonormal lattice filter algorithm.[7] For the array processing problem an optimum weighting vector is obtained by[6,8]

$$\underline{a} = \mu R^{-1} \underline{h} \tag{15}$$

where $h = [1 \;\; 0 \;\; 0 \;\; \ldots \;\; 0]'$ and μ is an arbitrary scalar. As in the Maximum Entropy Method, the power spectrum may be computed by[6]

$$P_W(\omega) = \frac{1}{|\underline{S}_\omega^\dagger \underline{a}|^2} = \frac{1}{\underline{S}_\omega^\dagger \underline{a}\,\underline{a}^\dagger \underline{S}_\omega} \quad . \tag{16}$$

Maximum Likelihood Method

The Maximum Likelihood method is based on filtering the data such that power at the frequency of interest is passed and all other frequency components are rejected in an optimal manner. In our notation, the power spectrum is given by [8,9,10]

$$P_{ML}(\omega) = \frac{1}{\underline{S}_\omega^\dagger R^{-1} \underline{S}_\omega} \quad . \tag{17}$$

Pisarenko Method

The Pisarenko method has not found as widespread use as the Wiener Filter method and the Maximum Likelihood method. Haykin[4] recently applied part of the Pisarenko method[11] to the array processing problem via a special autoregressive-moving average (ARMA) model. The complete Pisarenko method is based on a theorem of Caratheodory[12] that allows decomposition of the exact truncated covariance sequence r(n), $0 \le n \le q-1$, into a positively-weighted sum of q complex sinusoids and white noise. The method has three steps:

(i) identifying and removing the noise contribution to the covariance matrix,

(ii) forming the q×q covariance matrix R_q and and analysing the single eigenvector corresponding to the unique minimum eigenvalue $\lambda_{min} = 0$ to determine the sinusoid frequencies, and

(iii) solving a set of q simultaneous linear equations for the sinusoid powers.

Algebraic Processing Approach

We now formulate a generalized minimization problem which suggests several particular methods. Under different constraints, the solution of this problem encompasses each of the methods in the previous section.

Let us consider a general nontrivial p×1 coefficient vector $\underline{a}$ that is orthogonal to the noise-free component of each of the data vectors $\underline{y}_m$. From (12), this orthogonality is defined by the inner product relationship

$$0 = \langle \underline{a},\ \underline{y}_m - \underline{n}_m \rangle$$

$$= \sum_{k=1}^{q} A_k^* e^{-j\phi_{km}} \langle \underline{a},\ \underline{S}_{\omega_k} \rangle \quad , 1 \le m \le M \tag{18}$$

Since the $\{\omega_k\}$ are all different and the $\{\phi_{km}\}$ are random in nature, a little thought will convince oneself that $\underline{a}$ must be orthogonal to each of the q sinusoid vectors $\underline{S}_{\omega_k}$, $1 \le k \le q$. We next define the general z-transform A(z) of the coefficient vector $\underline{a}$ by

$$A(z) = \langle \underline{a},\ \underline{z}^* \rangle \tag{19}$$

where $\underline{z} = [1 \quad z^{-1} \quad z^{-2} \quad \ldots \quad z^{1-p}]'$. It is then readily shown that the orthogonality of $\underline{a}$ to each $\underline{S}_{\omega_k}$, $1 \le k \le q$, implies that A(z) must have q finite zeros located on the unit circle at the points $z_k = e^{j\omega_k}$, $1 \le k \le q$. With this in mind, the required frequencies can be determined by examination of the zeros of A(z).

In the idealized noise-free case, the snapshot vectors $\underline{y}_m$ are members of the q-dimensional subspace spanned by the q linearly independent sinusoid vectors $\underline{S}_{\omega_k}$, $1 \le k \le q$. Thus, for $q < p$ there always exists a p×1 vector $\underline{a}$ that is orthogonal to the noise-free components of the snapshot vectors $\underline{y}_m$. If $q \ge p$, there generally does not exist such a vector $\underline{a}$. Furthermore, when noise is present, even if $q < p$, there generally does not exist a vector $\underline{a}$ that is orthogonal to each of the noise-contaminated snapshot vectors $\underline{y}_m$. Nonetheless, it is intuitively desirable to select a coefficient vector which is nearly orthogonal to each of the snapshot vectors in some well-defined manner, and to determine the plane wave sinusoidal frequencies by examination of the zeros of the z-transform of this coefficient vector. A convenient method to evaluate these zero locations is to search for nulls in the magnitude of the Fourier transform of the coefficient vector. Since there can be more zeros than plane waves (i.e., $p-1>q$), we can estimate the $\{P_k\}$ in order to separate "signal zeros" from "noise zeros".

To obtain a mathematical measure of closeness to orthogonality, it is beneficial to define an orthogonality error vector $\underline{e}(\underline{a})$ whose m^{th} element is the inner product of $\underline{a}$ and $\underline{y}_m$, denoted by $\langle \underline{a}, \underline{y}_m \rangle$. We define the optimum $\underline{a}$ to be the vector $\underline{a}^\circ$ that minimizes some positive definite functional f of $\underline{e}(\underline{a})$. Hence we write

$$\underline{e}(\underline{a}) = [e(1) \quad e(2) \quad \ldots \quad e(M)]' \tag{20}$$

where

$$e(m) = \langle \underline{a}, \underline{y}_m \rangle$$

and

$$f(\underline{e}(\underline{a}^\circ)) = \min_{\underline{a} \in A} f(\underline{e}(\underline{a})) \tag{21}$$

where A is a constraint set for the solution vector $\underline{a}^\circ$. A constraint is generally necessary for the minimization to be well-defined, that is, for $\underline{a}^\circ$ to be unique and nontrivial.

We must choose an inner product for (20) and an error functional f for (21). Let us choose in particular the standard vector inner product $\langle \underline{a}, \underline{y}_m \rangle = \underline{a}' \underline{y}_m^*$ in which case we have

$$\underline{e}(\underline{a}) = \begin{bmatrix} \underline{a}'\underline{y}_1^* \\ \underline{a}'\underline{y}_2^* \\ \vdots \\ \underline{a}'\underline{y}_M^* \end{bmatrix} . \tag{22}$$

A convenient positive definite functional for an error vector is the mean square error criterion

$$f(\underline{e}) = E(\| \underline{e} \|^2) \tag{23}$$

where $\| \underline{e} \| = \sqrt{|e(1)|^2 + \ldots + |e(M)|^2}$, the Euclidean norm of $\underline{e}$.[14] It will be computationally expedient to normalize this criterion by the length M of the vector $\underline{e}$. From (22) and (23) we have

$$f(\underline{a}) = \frac{1}{M} E\left\{ \sum_{m=1}^{M} |\underline{a}'\underline{y}_m^*|^2 \right\}$$

$$= \underline{a}^\dagger R \underline{a} \tag{24}$$

where R is the covariance matrix defined in (14). Applying (21), the quadratic form (24) must now be minimized according to some constraint that causes $\underline{a}^\circ$ to be unique and nontrivial. Next we consider two possible constraints, a linear constraint and a quadratic constraint.

Linear Constraint

The first constraint is that $\underline{a}^\circ$ lies on a hyperplane specified by

$$A = \{a \in C^p : \underline{a}^\dagger \underline{h} + \underline{h}^\dagger \underline{a} = 2\} \tag{25}$$

where $\underline{h}$ is a nontrivial p×1 vector that characterizes the orientation of the hyperplane. The solution to (21) with this constraint is given by

$$\underline{a}^\circ = \left[\frac{1}{\underline{h}^\dagger R^{-1} \underline{h}}\right] R^{-1} \underline{h} \tag{26}$$

and the criterion's minimum value is

$$f(\underline{a}^\circ) = \frac{1}{\underline{h}^\dagger R^{-1} \underline{h}} \quad . \tag{27}$$

Quadratic Constraint

The second constraint is that $\underline{a}^\circ$ lies on the quadratic surface specified by

$$A = \{\underline{a} \in C^p : \underline{a}^\dagger W \underline{a} = 1\} \tag{28}$$

where W is a positive definite, Hermitian matrix which characterizes the quadratic surface. The solution to (21) with this constraint is given by

$$\underline{a}^\circ = \left[\frac{1}{\sqrt{\underline{x}_{min}^\dagger W \underline{x}_{min}}}\right] \underline{x}_{min} \tag{29}$$

and the criterion's minimum value is

$$f(\underline{a}^\circ) = \lambda_{min} \tag{30}$$

where $(\lambda_{min}, \underline{x}_{min})$ is the minimum-eigenvalue and eigenvector pair of $W^{-1}R$.

The above solutions encompass the three contemporary methods described earlier. For $\underline{h} = [1 \ 0 \ \ldots \ 0]'$, Equation (26) is precisely the Wiener Filter solution with $\mu = \frac{1}{\underline{h}^\dagger R^{-1} \underline{h}}$. Just as in linear prediction, this constraint implies that the first element of $\underline{a}^\circ$ is fixed at 1 and the other elements are unconstrained. For $\underline{h} = \underline{s}_\omega$, (27) is precisely the Maximum Likelihood solution.[3] This constraint requires $A^\circ(z)$ to have unity gain at $z = e^{j\omega}$, while the minimization strategy optimally reduces the gain at other frequencies. For $W = I_p$, the quadratic surface is the hypersphere of radius one, and (29) is a generalized version of the Pisarenko method. There are several differences. First, no special ARMA model is invoked, as is done by Haykin.[4] Second, neither noise power removal nor matrix order reduction are required, as they are in the Pisarenko method. Third, this method is based upon a minimization strategy and so justifies estimates, generally even non-Toeplitz, of the covariance matrix R. In the special case of a Toeplitz estimate matrix, a power identification technique similar to the Pisarenko method can be employed, as is shown later. Fourth, the general constraint matrix W allows greater flexibility in the solution. The quadratic constraint solution generalizes the Pisarenko method and extends it to the multiple-snapshot array processing problem.

Extension to Correlated Noise

Through selection of the constraint set, the algebraic approach extends quite readily to the case in which the contaminating noise is correlated. Such is the case when the noise is due not only to sensor measurement noise but also to a directional background noise field in the array environment. Note that any undesired signal (jamming interference, for example) may be considered as correlated noise.

Generalizing (14), we have that the data covariance matrix for the case of correlated noise is given by

$$R = \sigma^2 B + \sum_{k=1}^{q} \underline{s}_{\omega_k} \underline{s}_{\omega_k}^\dagger \tag{31}$$

where the noise covariance matrix B is defined by

$$\sigma^2 B = E\{\underline{n}_m \underline{n}_m^\dagger\} \ . \tag{32}$$

We assume that the shape of the noise spectrum is known, which implies knowledge of B.

Returning to the quadratic constraint (29,30), we note $(\lambda_{min}, \underline{x}_{min})$ is the solution to

$$(R - \lambda_{min} W)\underline{x}_{min} = \underline{0} \ . \tag{33}$$

We have seen that for the choice $W = I_p$ we have the Pisarenko solution. In this case it is well known that $\lambda_{min} = \sigma^2$ and that we essentially have a white-noise power cancellation algorithm.[13] Hence it is a simple step to choose

$$W = B \tag{34}$$

and achieve a colored-noise correlation cancellation algorithm. This step can be justified further by rewriting (31) as

$$(R - \sigma^2 B) = \sum_{k=1}^{q} \underline{s}_{\omega_k} \underline{s}_{\omega_k}^\dagger \tag{35}$$

and remembering that we seek a vector that is orthogonal to the sinusoid vectors $\{\underline{s}_{\omega_k}\}$. Also, it is apparent from the constraint $\underline{a}^\dagger B \underline{a} = 1$ that we are again specifying a set of vectors with constant norm, but now the norm is determined by the noise covariance matrix B.

The linear constraint (26,27) may also be extended for correlated noise. We shall only consider the Wiener solution, since it has been shown to achieve greater resolution than the Maximum Likelihood Method.[3,5] We will show later that to extend the Wiener linear prediction solution, a reasonable constraint $\underline{h}$ given B is

$$\underline{h} = B \begin{bmatrix} 1 \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix} \ . \tag{36}$$

Note that we no longer have a linear predictor.

Example

To illustrate the linear and quadratic constraint solutions, let us consider the case of a single plane wave of power P_1 and spatial frequency ω_1 incident on an array of two sensors in a correlated noise field. For this case the noise covariance matrix is given by

$$\sigma^2 B = \sigma^2 \begin{bmatrix} 1 & b^* \\ b & 1 \end{bmatrix} \tag{37}$$

where $|b|<1$, and the data vector covariance matrix by

$$R = \begin{bmatrix} P_1 + \sigma^2 & P_1 e^{-j\omega_1} + \sigma^2 b^* \\ P_1 e^{j\omega_1} + \sigma^2 b & P_1 + \sigma^2 \end{bmatrix}. \tag{38}$$

First we consider the linear and quadratic constraint solutions without accounting for the correlated noise (i.e., the Wiener and Pisarenko methods). For the linear constraint we choose $\underline{h} = [1 \;\; 0 \;\; \ldots \;\; 0]'$ and find from (26) that

$$\underline{a}^\circ = \begin{bmatrix} 1 \\ -\dfrac{P_1 e^{j\omega_1} + \sigma^2 b}{P_1 + \sigma^2} \end{bmatrix}. \tag{39}$$

Defining the signal-to-noise ratio by $SNR = P_1/\sigma^2$, we see that $A^\circ(z)$ has a zero located at

$$z = e^{j\omega_1}\left[\frac{SNR + b\, e^{-j\omega_1}}{SNR + 1}\right]. \tag{40}$$

As has been noted elsewhere,[15] this linear-prediction solution suffers from zero migration away from the unit circle even when the noise is white (i.e., $b = 0$). This migration degrades resolution, since applying the Fourier transform to evaluate zero locations may indicate only a single null when in fact there are two zeros close together somewhat off the unit circle. Furthermore, we see there is a frequency bias introduced by the correlated noise. This bias becomes greater as the signal-to-noise ratio decreases.

For the quadratic constraint we choose $W = I_p$ and find from (29) that

$$a^\circ = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -\dfrac{P_1 e^{j\omega_1} + \sigma^2 b}{|P_1 e^{j\omega_1} + \sigma^2 b|} \end{bmatrix}. \tag{41}$$

Thus $A^\circ(z)$ has a zero located at

$$z = \frac{SNR\, e^{j\omega_1} + b}{|SNR\, e^{j\omega_1} + b|}. \tag{42}$$

We see in this case that the zero lies directly on the unit circle, regardless of the signal-to-noise ratio. In fact, when the noise is white, the zero perfectly indicates the plane wave spatial frequency ω_1. However, when the noise is correlated, there is a frequency bias present that again becomes greater as the signal-to-noise ratio decreases.

Let us consider now the linear and quadratic constraint solutions which account for the correlated noise. For the linear constraint we choose $\underline{h} = B[1 \;\; 0 \;\; \ldots \;\; 0]'$ and find that

$$a^\circ = \mu \begin{bmatrix} 1 \\ -\dfrac{P_1(e^{j\omega_1} - b)}{P_1(1 - b\, e^{-j\omega_1}) + \sigma^2(1 - |b|^2)} \end{bmatrix} \tag{43}$$

where μ is a scalar function of P_1, ω_1, σ^2, and b. Thus $A^\circ(z)$ has a zero located at

$$z = e^{j\omega_1}\left[\frac{SNR(1 - b\, e^{-j\omega_1})}{SNR(1 - b\, e^{-j\omega_1}) + 1 - |b|^2}\right]. \tag{44}$$

As in (40), we see that pure white noise will still cause the zero to migrate away from the unit circle, and that correlated noise will introduce frequency bias. However, as the noise becomes "more correlated" (i.e., $|b| \to 1$), the zero moves closer to the unit circle and asymptotically indicates the exact plane wave spatial frequency ω_1, regardless of the signal-to-noise ratio. Note that the effect of an interfering harmonic source (i.e., $b = e^{j\omega_2}$) is completely removed.

For the quadratic constraint we choose $W = B$ and find that

$$a^\circ = \mu \begin{bmatrix} 1 \\ -e^{j\omega_1} \end{bmatrix} \tag{45}$$

where μ is a scalar function of ω_1 and b. Thus $A^\circ(z)$ has a zero located at

$$z = e^{j\omega_1}. \tag{46}$$

We see that the zero indicates the exact plane wave spatial frequency ω_1, regardless of the signal-to-noise ratio or the particular value of b. For this reason, we expect the quadratic-constraint solution to obtain high resolution.

To summarize the development to this point, the algebraic approach is based on approximating an orthogonality condition between a solution vector and each of the data vectors. This approach encompasses three contemporary array processing methods and readily extends to the case of correlated noise.

Implementation of the Quadratic-Constraint Solution

In the previous section we saw that the quadratic-constraint solution (29,30) is a promising array processing method in terms of its perfect resolution given exact covariance values. However, it requires an eigenvalue-eigenvector computation that seems to be quite burdensome. Fortunately, a simple recursive algorithm can be derived using the nature of the array processing problem.

First we recall the standard "inverse iteration"[16] method for finding the minimum eigenvalue and eigenvector pair of a complex matrix D. Consider the sequence of vectors $\{\underline{x}_k\}$ defined by

$$D\underline{x}_k = \underline{x}_{k-1} \tag{47}$$

where $\underline{x}_0$ is a nonzero and arbitrary. As k increases, we have

$$\underline{x}_k \to \underline{x}_{min} \quad \text{and} \quad \frac{\underline{x}_k^\dagger \underline{x}_{k-1}}{\underline{x}_k^\dagger \underline{x}_k} \to \lambda_{min} .$$

This method is appropriate to the array processing problem, in which the data arrives sequentially. Assume that from M snapshots we have estimated the covariance matrix by $\hat{R}_M$ and obtained the desired pair $(\lambda_M, \underline{x}_M)$. When the next snapshot is available, we form $\hat{R}_{M+1}$ and compute $\underline{x}_{M+1}$ from (47) using $D = \hat{R}_{M+1}$ and $\underline{x}_{k-1} = \underline{x}_M$. Since the inverse iteration method generally has fast convergence, a single iteration of (47) for $\underline{x}_{M+1}$ may be sufficient as long as $\hat{R}_M$ is only slowly time-varying. To accelerate convergence, we can apply the "inverse iteration of Wielandt"[16] wherein an approximation of λ_{min} is subtracted from the main diagonal of D before iterating. Given $\hat{R}_{M+1}$, we use λ_M to approximate λ_{M+1}. The iteration is given by

$$(\hat{R}_{M+1} - \lambda_M I)\underline{x}_{M+1} = \underline{x}_M ,$$

$$\underline{x}_{M+1} \to \underline{x}_{min} , \qquad \text{and}$$

$$\lambda_M + \frac{\underline{x}_{M+1}^\dagger \underline{x}_M}{\underline{x}_{M+1}^\dagger \underline{x}_{M+1}} = \lambda_{M+1} \to \lambda_{min} . \tag{48}$$

For a Toeplitz and Hermitian covariance matrix estimate, each iteration can be performed with $O(p^2)$ multiply-adds using Zohar's algorithm.[17] An alternate algorithm by Gueguen[18] can be used to avoid numerical difficulties that may be associated with $(\hat{R}_{M+1} - \lambda_M I)$.

For the case of correlated noise, we have $D = B^{-1}\hat{R}_{M+1}$. We suggest generalizing the accelerated iteration to avoid calculation of B^{-1}. Our general iteration is given by

$$(\hat{R}_{M+1} - \lambda_M B)\underline{x}_{M+1} = \underline{y}_M = B\underline{x}_M ,$$

$$\underline{x}_{M+1} \to \underline{x}_{min} , \qquad \text{and}$$

$$\lambda_M + \frac{\underline{x}_{M+1}^\dagger \underline{y}_M}{\underline{x}_{M+1}^\dagger \underline{y}_{M+1}} = \lambda_{M+1} \to \lambda_{min} . \tag{49}$$

Each iteration still only requires $O(p^2)$ multiply-adds, and B^{-1} is never calculated.

In some applications, it may not be desirable to calculate the solution vector after every snapshot. For instance, forming a new covariance matrix estimate and calculating a new solution only every L snapshots reduces the average computation rate by a factor of L. Unfortunately, if the new covariance matrix estimate differs considerably from the previous, the previous eigenvalue may be a poor approximation to λ_{min}, and convergence will be slowed. To obtain a better eigenvalue approximation, we apply perturbation techniques.[19] Suppose that D and W are Hermitian matrices and that we have solved the eigenvalue-eigenvector problem

$$D\underline{x} = \lambda W\underline{x} . \tag{50}$$

Applying a Hermitian perturbation δD to D we have the new problem

$$(D + \delta D)(\underline{x} + \delta\underline{x}) = (\lambda + \delta\lambda)W(\underline{x} + \delta\underline{x}) . \tag{51}$$

It can be shown that $\delta\lambda$ is approximately given by

$$\delta\lambda \doteq \frac{\underline{x}^\dagger(\delta D)\underline{x}}{\underline{x}^\dagger W\underline{x}} \tag{52}$$

Our approximation to the new λ_{min} is then given by the sum of $\delta\lambda$ and the previous λ_{min}. With appropriate definitions, this approximation replaces λ_M in (49) when we expect the new covariance matrix estimate to differ considerably from the previous estimate.

Relationship to Linear-Constraint Solution

The iterative implementation of the quadratic-constraint solution gives insight into the linear-constraint solution. Namely, the first iteration of (47) with $D = R$ and $\underline{x}_0 = \underline{h}$ yields the linear-constraint solution of (26) within a constant of proportionality. Repeated calculation of the linear-constraint solution, with $\underline{h}$ at each step equal to $\underline{a}^\circ$ of the previous step, is in fact an iterative implementation of the quadratic-constraint solution. It is apparent that at each step, the constraining hyperplane is realigned according to the estimated solution. With these insights, it is reasonable to choose

$$\underline{h} = B \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{53}$$

as the linear constraint for correlated noise, since it yields the first step of the iterative quadratic-constraint solution (without acceleration) for correlated noise given in (49). This justifies the choice made earlier in (36).

In this section we have presented a recursive algorithm (49) for implementing the quadratic-constraint solution. The algorithm makes use of the sequential nature of the snapshot data to efficiently employ inverse iteration. The algorithm includes the case of correlated noise. A modification to the algorithm (52) was presented for the case where successive covariance matrix estimates differ considerably.

Covariance Matrix Estimate

To employ the proposed processing methods, an estimate of the covariance matrix is required. From this estimate, a solution vector is obtained and the zeros of the vector's z-transform examined to determine the plane wave spatial frequencies. Given a p×1 solution vector and q plane waves, q<p, there will be q "signal" zeros and p-q-1 "noise" zeros. These zeros must be separated from one another. It is well known that in the linear prediction solution, dominant frequency components will generate zeros closer to the unit circle than less powerful components; thus, a simple way to evaluate signal zero locations is to search for nulls in the solution vector's Fourier transform. For the quadratic solution in white noise, it can be shown[20] that all of the zeros will be on the unit circle when the covariance matrix estimate is both Hermitian and Toeplitz. Thus the estimated frequencies can be directly employed in a power determination technique[11,13] and the zeros separated on a basis of signal power as before.

A standard covariance matrix estimate[3] is

$$\hat{R}_M = \frac{1}{M}\sum_{m=1}^{M} \underline{y}_m \underline{y}_m^{\dagger} \quad . \tag{53}$$

This estimate is unbiased, Hermitian, but in general not Toeplitz. Furthermore, only one lag product from each data vector is used in formulating each element of $\hat{R}_M$. An alternate estimate is the matrix $\tilde{R}_M$ whose elements are given by

$$\tilde{R}_M(i,j) = c(i-j), \quad 1 \le i,j \le p \tag{54}$$

where

$$c(n) = \frac{1}{M}\sum_{m=1}^{M} \frac{1}{p-n} \sum_{\ell=0}^{p-n-1} y_m(\ell+n) y_m^*(\ell), \quad 0 \le n \le p-1$$

$$c(n) = c^*(-n) \quad , \quad -p+1 \le n < 0.$$

This estimate is unbiased, Hermitian, and Toeplitz. Also, p-n lag products from each data vector are used in formulating each element c(n). Thus the estimate of (54) has lower variance than that of (53).

In the following simulations, the standard non-Toeplitz estimate $\hat{R}_M$ (53) will be used in the linear-constraint solution in order to compare with previous simulations.[3] However, for the quadratic-constraint solution the Toeplitz structure is important, hence the Toeplitz estimate $\tilde{R}_M$ (54) will be used.

Simulation Results

To compare the performance of these two processing methods, the data vectors (12) were generated by computer simulation. The simulation model corresponded to that chosen by Gabriel[3] in his comparative paper. Namely, the case of two sources incident on an array with white noise was considered. The parameter selections were $q = 2$, $p = 8$, $\sigma^2 = 1$, $A_1 = A_2 = 31.62$ (30 dB SNR) and 3.162 (10 dB SNR), $\theta_1 = 18°$, $\theta_2 = 22°$, $M = 50$ (many snapshots) and 10 (few snapshots), and $d = \lambda/2$.

With white noise, the linear solution and the quadratic solution correspond to the Wiener and Pisarenko methods. The simulation results are shown in Figure 3. In this figure, the linear solution has been evaluated via its Fourier transform and the quadratic solution via the power determination technique. Overlayed solutions for ten different realizations of the random data are shown to give a sense of each method's consistency.

These results show that both methods work well at high SNR with many snapshots. However, the linear solution performs poorly at low SNR with few snapshots, while the quadratic solution continues to give good resolution and good suppression of spurious effects. In general, the quadratic solution has shown better performance than the linear solution over a wide range of conditions.[20] Further simulations are underway to compare the performance of the methods in correlated noise and to evaluate the recursive algorithm presented above. The results will be given at the conference.

Conclusions

We have detailed an algebraic approach to array processing based upon approximation of an orthogonality condition. The approach encompasses several contemporary, high resolution methods. Previous results were extended to the case of correlated noise, and a recursive algorithm presented for the quadratic-constraint solution. The quadratic-constraint solution appears to be particularly effective and suggests further investigation of eigen-analysis array processing methods and their implementation.

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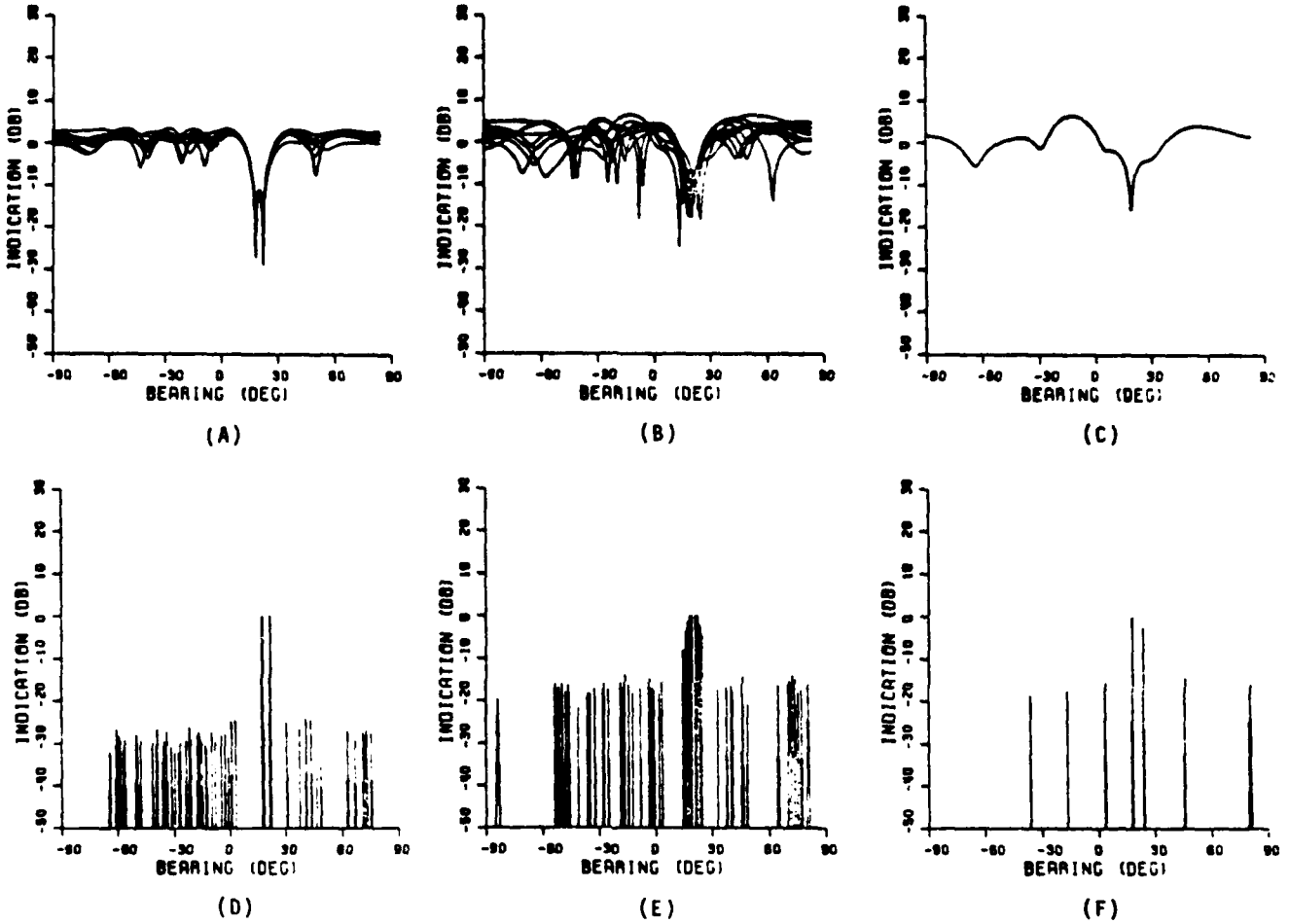

Figure 3. Two-source simulation with sources at 18 and 22 degrees.

(A) 30 dB SNR, 50 snapshots
(B) 10 dB SNR, 10 snapshots } Linear solution
(C) Single trial from (B)

(D) 30 dB SNR, 50 snapshots
(E) 10 dB SNR, 10 snapshots } Quadratic solution
(F) Single trial from (E)

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In this report, an algebraic characterization of the problem of resolving closely spaced plane waves incident on a linear array is developed. This characterization describes many well-known techniques for solving this problem as well as introducing a new effective approach.

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